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# Equivalence transformations and approximate solutions of a nonlinear heat conduction model

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**Abstract.** We propose a method of finding approximate solutions associated with nonlinear partial differential equations involving a small parameter by using equivalence transformations. As an application of our method we consider a one-dimensional nonlinear heat conduction model and study its approximate solutions.

## 1. Introduction

In recent years considerable progress has been made in finding particular solutions of nonlinear partial differential equations (PDEs) which arise in several branches of physics, mathematics, engineering and biology through Lie group analysis [1–7]. Recently several new approaches have been proposed. In particular, some techniques have been developed to find approximate solutions.

Quite often the nonlinear PDEs depend on some small parameters and in the classical Lie group analysis these parameters have been treated as constants. However, to study such evolutionary equations with small parameters an interesting new concept, approximate transformation groups, has been developed by Baikov *et al* (see e.g. [5, 8] and references therein) in order to find approximately invariant solutions.

The main motivation of this paper is to generate approximate solutions of a given nonlinear PDE involving a small parameter not in the framework of approximate symmetries but through exact equivalence transformation groups. The starting point of our procedure is an example given by Baikov *et al* to illustrate the use of equivalence transformations for constructing approximately invariant solutions (see [5, vol 3, p 61]). As an application we consider a one-dimensional hyperbolic heat conduction model introduced in [9, 10] and successively studied in [11, 12]. We consider a class of systems parametrized by the relaxation time, $z$, and find a group of equivalence transformations of this class of systems whose Lie algebra is an infinite-dimensional algebra. By using the invariance surface condition related to the equivalence transformations we reduce the system under consideration to a new system of PDEs (hereafter we call it RS) where the new independent variables are similarity variables of the equivalence transformations. Under suitable hypothesis, by expanding the dependent variables of RS in series, we find some approximate solutions.

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We wish to stress that this approach is, of course, different from other approximate analysis introduced by Fushchych and Shtelen [13] and subsequently developed by Euler *et al* [14]. In fact, in the framework of the theory proposed by Fushchych and Shtelen, an approximate symmetry of order $n$ for an assigned equation is defined as an exact symmetry of the approximate system obtained by expanding the field variable with respect to the small parameter, truncating the series of order $n$, and equating to zero the coefficients of the powers of the small parameter. Consequently, they found exact solutions of the reduced system which corresponds to approximate solutions of the original equation.

The plan of this paper is as follows. In section 2 we present a short sketch of a mathematical model for heat conduction. In section 3 we briefly discuss the Lie symmetries of the Fourier model and hyperbolic model for heat conduction. In section 4 we find equivalence transformations associated with the heat conduction model. In section 5 we describe the method of constructing approximate solutions from the equivalence transformations. In this section classes of approximate solutions found are as in section 6. Conclusions are presented in section 7.

## 2. Short sketch of a mathematical model for heat conduction

The following equation has been proposed [9, 10] to describe the heat conduction in a homogeneous isotropic rigid body:

$$z\boldsymbol{q}_t + (1 - zL'L^{-1}T_t)\boldsymbol{q} + L \bigtriangledown T = 0. \tag{2.1}$$

In the above equation, $T, \boldsymbol{q}, z$ represent the absolute temperature, heat flux and a positive relaxation constant, respectively; subscripts denote partial differentiation with respect to the variable $t$, and a prime denotes the derivative with respect to $T$. We denote $L = L(T)$ the thermal conductivity. Equation (2.1) removes the well known paradox of infinite speed of heat propagation and is a generalization of the Cattaneo–Maxwell equation introduced in their well known works [15–17]. Equation (2.1) gives the Fourier laws and Cattaneo's equation for $z = 0$ and $L =$ constant, respectively, as subcases.

From the extended thermodynamics (ET) (see e.g [18–20]) point of view one can consider equation (2.1) as a field equation which is associated, as usual, with the following energy conservation equation:

$$e_t + \bigtriangledown \cdot \boldsymbol{q} = 0 \tag{2.2}$$

where it is assumed, in agreement with ET, that $e = e(\boldsymbol{q}, T)$. It is also possible to verify the thermodynamic compatibility of the model described by equations (2.1) and (2.2) for heat propagation in the framework of ET.

In fact, in [11, 12], starting from [9, 10], after observing that equation (2.1) can be written in a conservative form, we found that there exists a supplementary conservation law of the form

$$h_t + \bigtriangledown \cdot \boldsymbol{J} = r \tag{2.3}$$

with $h = h(\boldsymbol{q}^2, T)$ and $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{q}, T)$ which, by requiring $r \geqslant 0$, is equivalent to a generalized form of the entropy inequality [18–20].

The procedure used in [11] gives the following characterization of the functional form of $e, h$ and $\boldsymbol{J}$:

$$\begin{aligned} e &= e_0(T) + \frac{z}{2L}\left(\frac{2}{T} - \frac{L'}{L}\right)\boldsymbol{q}^2 \\ h &= h_0(T) + \frac{z}{2LT}\left(\frac{1}{T} - \frac{L'}{L}\right)\boldsymbol{q}^2 \\ \boldsymbol{J} &= \frac{\boldsymbol{q}}{T} \end{aligned} \tag{2.4}$$

where $e_0(T)$ is the internal energy at equilibrium and $h_0' = C/T$ ($C := e_0'(T)$, the positive specific heat at equilibrium). Moreover, the request that $r \geqslant 0$ implies, as expected, $L \geqslant 0$. Furthermore, it is shown in [11] that it is possible to write the system (2.1) and (2.2) as a symmetric quasilinear hyperbolic system [21, 22] so that a general theorem of well-posedness of the Cauchy problem holds [23].

Finally, it is worth mentioning that in [19, 20], the interested reader can find not only theoretical arguments for models of heat-wave propagation based on the Maxwell–Cattaneo model equation but also find a vast literature for experimental backing for these models.

So taking into account equation (2.4) the governing system for heat propagation can be written as

$$\begin{aligned} &z\boldsymbol{q}_t + (1 - zL'L^{-1}T_t)\boldsymbol{q} + L \bigtriangledown T = \boldsymbol{0} \\ &\left(C + \frac{z}{2}A'\boldsymbol{q}^2\right)T_t + zA\boldsymbol{q}\cdot\boldsymbol{q}_t + \bigtriangledown\cdot\boldsymbol{q} = 0 \end{aligned} \tag{2.5}$$

where

$$A := \frac{1}{L}\left(\frac{2}{T} - \frac{L'}{L}\right). \tag{2.6}$$

In this paper we restrict ourselves to the unidimensional case and assume $L = L_0T^2$ and $C = \bar{C}T^3$. For this form of $L(T)$, $A$ vanishes and our system reads

$$\begin{aligned} &T_x - \frac{2zq}{L_0T^3}T_t + \frac{z}{L_0T^2}q_t + \frac{1}{L_0T^2}q = 0 \\ &q_x + \bar{C}T^3T_t = 0. \end{aligned} \tag{2.7}$$

This can also be obtained as a special case for a rigid heat conductor model introduced in [24].

For further developments we rewrite the system (2.7) in nondimensional form by introducing the variables

$$\hat{x} = \frac{x}{\bar{x}} \qquad \hat{t} = \frac{t}{\bar{t}} \qquad \hat{u} = \frac{T}{\bar{T}_0} \qquad \hat{v} = \frac{q\bar{x}}{L_0\bar{T}_0^3} \qquad \hat{z} = \frac{z}{\bar{t}} \tag{2.8}$$

where $\bar{t}, \bar{x}, \bar{T}$ are characteristic time, length and temperature. If $\bar{t}$ represents a macroscopic timescale: $z \ll \bar{t}$ then $\hat{z} \ll 1$. So our system, after dropping the hat and putting it in evolutionary form, reads

$$\begin{aligned} &zv_t - \frac{2zv}{u}u_t + u^2u_x + v = 0 \\ &u_t + \frac{1}{\hat{C}u^3}v_x = 0 \end{aligned} \tag{2.9}$$

where we put

$$\hat{C} = \frac{\bar{C}\bar{T}_0\bar{x}^2}{L_0\bar{t}}. \tag{2.10}$$

It is a simple matter to see that when $z = 0$, system (2.9) becomes

$$\begin{aligned} &u^2 u_x + v = 0 \\ &u_t + \frac{1}{\hat{C}u^3} v_x = 0 \end{aligned} \tag{2.11}$$

which fall in the case of a Fourier conduction model which is reducible to the following nonlinear diffusion equation

$$u_t = \frac{1}{\hat{C}u^3}(u^2 u_x)_x. \tag{2.12}$$

Hereafter, for the sake of simplicity, we call system (2.9) a *perturbed system* and (2.12) an *unperturbed system*.

## 3. Lie symmetries of the unperturbed and perturbed systems

Before going on to study the equivalence transformations of the perturbed nonlinear heat conduction model, equation (2.9), we briefly review the Lie symmetries of the unpertubed system (2.11).

Applying the classical Lie algorithm [1–7] to equation (2.11), we get the following infinitesimal components for the generator $\Gamma$ of the systems (2.11) by using the software package of [25]:

$$\xi_1 = (-3a_1 + a_2)x + a_3 \qquad \xi_2 = (2a_2 - 7a_1)t - a_4 \qquad \phi_1 = -a_1 u \qquad \phi_2 = -a_2 v \tag{3.1}$$

where $\xi_1, \xi_2, \phi_1$, and $\phi_2$ are the infinitesimal components corresponding to the variables $x, t, u$ and $v$ and $a_1, a_2, a_3$ and $a_4$ are arbitrary constants.

The basis of the associated Lie algebra $L_4$ is

$$\begin{aligned} &\Gamma_1 = \frac{\partial}{\partial x} \qquad \Gamma_2 = \frac{\partial}{\partial t} \qquad \Gamma_3 = -3x\frac{\partial}{\partial x} - 7t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} \\ &\Gamma_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}. \end{aligned} \tag{3.2}$$

It is a simple matter, by specializing the results obtained in [26], to obtain the following symmetries for the perturbed system:

$$X_1 = \frac{\partial}{\partial x} \qquad X_2 = \frac{\partial}{\partial t} \qquad X_3 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 7v\frac{\partial}{\partial v}. \tag{3.3}$$

Moreover we can verify that

$$X_3 = 2\Gamma_3 + 7\Gamma_4. \tag{3.4}$$

Therefore, the three-dimensional algebra $L_3 = \Gamma_1 \oplus \Gamma_2 \oplus (2\Gamma_3 + 7\Gamma_4)$ is a *stable* subalgebra, in the sense of [5], of the unperturbed system.

## 4. Equivalence transformations

Let us consider a system of PDEs. An equivalence transformation is a transformation which changes the system into another system, having the same differential structure, but with a different form of coefficients. Quite often the equivalence transformations can be found for the systems when the constitutive elements are arbitrary functions. The search of continuous equivalence transformations can be done either through direct approach or by

Lie's infinitesimal criterion orginally suggested by Ovsiannikov [3]. Even though the direct approach, in principle, gives the complete group transformation, it involves very lengthy calculations and so quite often one uses Lie's infinitesimal criterion to find the equivalence transformations. The main advantage of this is that it offers a practical way to find the equivalence transformations even though it does not give the largest group of equivalence transformations [27, 28].

As far as our system is concerned, even though it does not contain any arbitrary function, it involves a numerical parameter $z$ characterizing the type of conductor. When we consider different types of conductors the value of $z$ changes, so we can treat $z$ as a parameter describing the class of systems having the form (2.9). An equivalence transformation for the system (2.9) changes the original equation into a new equation having the same differential structure with a different value of the parameter. In other words, the equivalence transformation changes the function $z = z_1 \equiv$ constant into another function $z = z_2 \equiv$ constant, in general with $z_1 \neq z_2$. If $z_1 = z_2$, then the equivalence transformation is a symmetry transformation of the system (2.9). However, it is easy to verify that one can bring out this kind of equivalence transformation from Lie's infinitesimal criterion itself by treating the parameter as a new independent variable so that the field variables will be considered as

$$\begin{aligned} u &= u(t, x, z) \\ v &= v(t, x, z). \end{aligned} \tag{4.1}$$

To carry out our investigations let us rewrite equation (2.9) in the following form:

$$\begin{aligned} &u_t + \frac{v_x}{\hat{C}u^3} = 0 \\ &zv_t + u^2 u_x + \frac{2zv}{\hat{C}u^4} v_x + v = 0. \end{aligned} \tag{4.2}$$

The invariance of the equation (4.2) under the following local one-parameter Lie group of infinitesimal point transformations,

$$x^* = x + \varepsilon\xi_1(t, x, z) \tag{4.3a}$$
$$t^* = t + \varepsilon\xi_2(t, x, z) \tag{4.3b}$$
$$z^* = z + \varepsilon\xi_3(t, x, z, u, v) \tag{4.3c}$$
$$u^* = u + \varepsilon\phi_1(t, x, z, u, v) \tag{4.3d}$$
$$v^* = v + \varepsilon\phi_2(t, x, z, u, v) \tag{4.3e}$$

leads to the following determining equations

$$\xi_{3u} = \xi_{3v} = \xi_{3x} = \xi_{3t} = 0 \tag{4.4a}$$
$$\frac{z}{\hat{C}u^3}\phi_{2u} - u^2\phi_{1v} + \frac{1}{\hat{C}u}\xi_{2x} - z\xi_{1t} = 0 \tag{4.4b}$$
$$\frac{z}{\hat{C}u^3}\phi_{2x} + \frac{v}{\hat{C}u^3}\xi_{2x} - v\phi_{1v} + z\phi_{1t} = 0 \tag{4.4c}$$
$$\frac{2v}{\hat{C}u^4}\xi_{2x} - \frac{2v}{u}\phi_{1v} + \phi_{2v} - \phi_{1u} + \xi_{2t} - \xi_{1x} - \frac{3}{u}\phi_1 = 0 \tag{4.4d}$$
$$\frac{2zv}{\hat{C}u^2}\xi_{2x} + zu^2\xi_{2t} - zu^2\xi_{1x} + zu^2\phi_{1u} + \frac{2z^2v}{\hat{C}u^4}\phi_{2u} - zu^2\phi_{2v} - u^2\xi_3 + 2zu\phi_1 = 0 \tag{4.4e}$$
$$\frac{4zv^2}{\hat{C}^2u^8}\xi_{2x} - \frac{2zv}{\hat{C}u^4}\xi_{1x} + \frac{1}{\hat{C}u}\xi_{2x} + \frac{2zv}{\hat{C}u^4}\xi_{2t} + u^2\phi_{1v} - \frac{z}{\hat{C}u^3}\phi_{2u} - \frac{8zv}{\hat{C}u^5}\phi_1 - z\xi_{1t} + \frac{2z}{\hat{C}u^4}\phi_2 = 0 \tag{4.4f}$$

$$\frac{2zv^2}{\hat{C}u^4}\xi_{2x} + zv\xi_{2t} + zu^2\phi_{1x} + \frac{2z^2v}{\hat{C}u^4}\phi_{2x} - zv\phi_{2v} + z^2\phi_{2t} - v\xi_3 + z\phi_2 = 0. \tag{4.4g}$$

Solving equation (4.4*a*–*g*) consistently we obtain the following solution:

$$\begin{aligned} &\xi_1 = z(a_1(z)x + a_2(z)) \qquad \xi_2 = a_3(z)t + za_4(z) \qquad \xi_3 = za_3(z) \\ &\phi_1 = (-2za_1(z) + a_3(z))u \qquad \phi_2 = (3a_3(z) - 7za_1(z))v \end{aligned} \tag{4.5}$$

where $a_1$, $a_2$, $a_3$ and $a_4$ are arbitrary functions of $z$.

The associated vector field can be written as

$$\begin{aligned} Y = {}& z(a_1(z)x + a_2(z))\frac{\partial}{\partial x} + (a_3(z)t + za_4(z))\frac{\partial}{\partial t} + za_3(z)\frac{\partial}{\partial z} + (-2za_1(z) + a_3(z))u\frac{\partial}{\partial u} \\ &+ (3a_3(z) - 7za_1(z))v\frac{\partial}{\partial v}. \end{aligned} \tag{4.6}$$

From $Y$ we can get equivalence generators *projectable* in the space $(t, x, u, v)$ by specializing the arbitrary functions as follows:

$$a_1(z) = \frac{\mu_1}{z} \qquad a_2(z) = \frac{\mu_2}{z} a_3(z) = \mu_3 \qquad a_4(z) = \frac{\mu_4}{z} \tag{4.7}$$

where $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ are arbitrary constants. Therefore, the equivalence projectable generator $Y_P$ is of the form

$$Y_P = (\mu_1 x + \mu_2)\frac{\partial}{\partial x} + (\mu_3 t + \mu_4)\frac{\partial}{\partial t} + \mu_3 z\frac{\partial}{\partial z} + (\mu_3 - 2\mu_1)u\frac{\partial}{\partial u} + (3\mu_3 - 7\mu_1)v\frac{\partial}{\partial v}. \tag{4.8}$$

By projecting this generator into the space $(t, x, u, v)$ we can easily check that it coincides with the four-dimensional Lie algebra spanned by symmetries of the unperturbed system.

## 5. Approximate solutions

In this section we show how to find approximate solutions for system (4.2) from the equivalence transformations (4.5).

The invariance surface condition associated with the projectable generator, $Y_P$, (4.8) can be written as

$$(\mu_1 x + \mu_2)\frac{\partial u}{\partial x} + (\mu_3 t + \mu_4)\frac{\partial u}{\partial t} + \mu_3 z\frac{\partial u}{\partial z} = (\mu_3 - 2\mu_1)u \tag{5.1a}$$

$$(\mu_1 x + \mu_2)\frac{\partial v}{\partial x} + (\mu_3 t + \mu_4)\frac{\partial v}{\partial t} + \mu_3 z\frac{\partial v}{\partial z} = (3\mu_3 - 7\mu_1)v. \tag{5.1b}$$

The general solution of the above equations, (5.1), can be found by solving the characteristic equation

$$\frac{\mathrm{d}x}{(\mu_1 x + \mu_2)} = \frac{\mathrm{d}t}{(\mu_3 t + \mu_4)} = \frac{\mathrm{d}z}{\mu_3 z} = \frac{\mathrm{d}u}{(\mu_3 - 2\mu_1)u} = \frac{\mathrm{d}v}{(3\mu_3 - 7\mu_1)v}. \tag{5.2}$$

In the case $\mu_1, \mu_3 \neq 0$ (we will discuss the case $\mu_1, \mu_3 = 0$ separately in the next section) we get,

$$u = \psi(\sigma, \eta)(\mu_3 t + \mu_4)^{\frac{(\mu_3 - 2\mu_1)}{\mu_3}} \qquad v = \chi(\sigma, \eta)(\mu_3 t + \mu_4)^{\frac{(3\mu_3 - 7\mu_1)}{\mu_3}} \tag{5.3a}$$

and

$$\sigma = \frac{(\mu_1 x + \mu_2)^{\mu_3}}{(\mu_3 t + \mu_4)^{\mu_1}} \qquad \eta = \frac{z}{(\mu_3 t + \mu_4)}. \tag{5.3b}$$

By substituting (5.3) in (4.2) we get a reduced system (RS) of PDEs in $\psi$ and $\chi$, where the independent variables are now $\sigma$ and $\eta$, of the form

$$\begin{aligned}
&\mu_1\mu_3\sigma\frac{\partial\psi}{\partial\sigma}+\mu_3\eta\frac{\partial\psi}{\partial\eta}-\frac{\mu_1\mu_3}{\hat{C}\psi^3}\sigma^{\frac{\mu_3-1}{\mu_3}}\frac{\partial\chi}{\partial\sigma}-(\mu_3-2\mu_1)\psi=0\\
&\mu_1\mu_3\sigma\eta\frac{\partial\chi}{\partial\sigma}+\mu_3\eta^2\frac{\partial\chi}{\partial\eta}-\mu_1\mu_3\psi^2\sigma^{\frac{\mu_3-1}{\mu_3}}\frac{\partial\psi}{\partial\sigma}-\frac{2\eta\mu_1\mu_3\chi}{\hat{C}\psi^4}\sigma^{\frac{\mu_3-1}{\mu_3}}\frac{\partial\chi}{\partial\sigma}\\
&\qquad -(1+\eta(3\mu_3-7\mu_1))\chi=0.
\end{aligned}\tag{5.4}$$

Once observed that $\eta$ is (at least) locally of the same order of $z$ and assuming that $\psi$ and $\chi$ are analytic functions of its arguments, it is possible to expand them in series so that we can write

$$\begin{aligned}
\psi&=\psi_0(\sigma)+\eta\psi_1(\sigma)+\eta^2\psi_2(\sigma)+\cdots\\
\chi&=\chi_0(\sigma)+\eta\chi_1(\sigma)+\eta^2\chi_2(\sigma)+\cdots.
\end{aligned}\tag{5.5}$$

By truncating (5.5) at order $n$, substituting them in the RS, neglecting the higher powers of $\eta$ and by equating the coefficients of various powers of $\eta^i$, $i=0,\ldots,n$, to zero in both equations we get a system of $2n+2$ ordinary differential equations (ODEs) in the unknowns $\psi_0,\ldots,\psi_n$, $\chi_0,\ldots,\chi_n$, with the independent variable $\sigma$. In this system only the subsystem of equations in the unknowns $\psi_0$ and $\chi_0$ is nonlinear and it is decoupled from the other $2n$ equations. It is easy to check that this subsystem is the reduced system of the unperturbed system by means of the projection of $Y_P$ to the space of $(t,x,u,v)$. Once this nonlinear system is solved, that is, once we have found an invariant solution of the unpertubed system, the remaining equations can be reduced to a linear form which, in general, can be solved easily.

By substituting equation (5.5) in (5.4) with the first-order approximation in $\eta$, we get

$$\begin{aligned}
&\mu_1\mu_3\sigma(\psi_0^3+3\eta\psi_0^2\psi_1)(\psi_0'+\eta\psi_1')+\mu_3\eta(\psi_0^3+3\eta\psi_0^2\psi_1)\psi_1\\
&\qquad-\frac{\mu_1\mu_3}{\hat{C}}\sigma^{\frac{(\mu_3-1)}{\mu_3}}(\chi_0{}'+\eta\chi_1')-(\mu_3-2\mu_1)(\psi_0^4+4\eta\psi_0^3\psi_1)=0\\
&\mu_1\mu_3\sigma\eta(\psi_0^4+4\eta\psi_0^3\psi_1)(\chi_0'+\eta\chi_1')-\frac{2\eta\mu_1\mu_3}{\hat{C}}\sigma^{\frac{(\mu_3-1)}{\mu_3}}(\chi_0+\eta\chi_1)(\chi_0{}'+\eta\chi_1')\\
&\qquad-\mu_1\mu_3(\psi_0^6+6\eta\psi_0^5\psi_1)\sigma^{\frac{(\mu_3-1)}{\mu_3}}(\psi_0{}'+\eta\psi_1')+\mu_3\eta^2(\psi_0^4+4\eta\psi_0^4\psi_1)\chi_1\\
&\qquad-(1+\eta(3\mu_3-7\mu_1))(\chi_0+\eta\chi_1)(\psi_0^4+4\eta\psi_0^3\psi_1)=0
\end{aligned}\tag{5.6}$$

where the prime denotes differentiation with respect to $\sigma$. Equating the coefficients of $\eta^i$, $i=0,1$ to zero and neglecting the higher powers of $\eta$ we get

$$\mu_1\mu_3\sigma\psi_0^3\psi_0'-\frac{\mu_1\mu_3}{\hat{C}}\sigma^{\frac{\mu_3-1}{\mu_3}}\chi_0'-(\mu_3-2\mu_1)\psi_0^4=0\tag{5.7$a$}$$

$$\mu_1\mu_3\sigma^{\frac{\mu_3-1}{\mu_3}}\psi_0^2\psi_0'+\chi_0=0\tag{5.7$b$}$$

$$\mu_1\mu_3\sigma\psi_0^3\psi_1'+3\mu_1\mu_3\sigma\psi_0^2\psi_1\psi_0'-\frac{\mu_1\mu_3}{\hat{C}}\sigma^{\frac{\mu_3-1}{\mu_3}}\chi_1'+(8\mu_1-3\mu_3)\psi_0^3\psi_1=0\tag{5.7$c$}$$

$$\begin{aligned}
&\mu_1\mu_3\sigma^{\frac{\mu_3-1}{\mu_3}}\psi_0^6\psi_1'-\mu_1\mu_3\sigma\psi_0^4\chi_0'+6\mu_1\mu_3\sigma^{\frac{\mu_3-1}{\mu_3}}\psi_0^5\psi_1\psi_0'+\frac{2\mu_1\mu_3}{\hat{C}}\sigma^{\frac{(\mu_3-1)}{\mu_3}}\chi_0'\chi_0\\
&\qquad+\psi_0^4\chi_1+(3\mu_3-7\mu_1)\psi_0^4\chi_0+4\psi_0^3\psi_1\chi_0=0.
\end{aligned}\tag{5.7$d$}$$

From equations (5.7$a$, $b$) we get a particular solution of the form

$$\psi_0=\frac{14\mu_1^2}{\hat{C}\mu_3\sigma^{(2/\mu_3)}}\tag{5.8}$$

$$\chi_0 = \frac{5488\mu_1^7}{\hat{C}^3\mu_3^3\sigma^{(7/\mu_3)}}. \tag{5.9}$$

Substituting equations (5.8) and (5.9) in (5.7$c$) and (5.7$d$) and simplifying we get the following linear differential equations for $\psi_1$, $\chi_1$:

$$\frac{2744\mu_1^6}{\hat{C}^2\mu_3^2\sigma^{(6/\mu_3)-1}}\psi_1' - \mu_3\sigma^{\frac{(\mu_3-1)}{\mu_3}}\chi_1' - \frac{2744(3\mu_3-2\mu_1)\mu_1^5}{\hat{C}^2\mu_3^3\sigma^{(6/\mu_3)}}\psi_1 = 0 \tag{5.10}$$

$$\frac{196\mu_1^5}{\hat{C}^2\mu_3\sigma^{(5/\mu_3)-1}}\psi_1' - \frac{784\mu_1^5}{\hat{C}^2\mu_3^2\sigma^{(5/\mu_3)}}\psi_1 + \chi_1 + \frac{5488\mu_1^7}{\hat{C}^3\mu_3^2\sigma^{(7/\mu_3)}} = 0. \tag{5.11}$$

Solving equations (5.10) and (5.11), we get

$$\psi_1 = \frac{I_1}{\sigma^{(2/\mu_3)}} + \frac{I_2}{\sigma^{\frac{(14\mu_1-11\mu_3)}{\mu_3^2}}} + \frac{196\mu_1^2}{\hat{C}(14\mu_1-13\mu_3)\sigma^{(2/\mu_3)}}\log\sigma \tag{5.12}$$

$$\chi_1 = \frac{10\,976(3\mu_3-7\mu_1)\mu_1^7}{\hat{C}^3\mu_3^2(14\mu_1-13\mu_3)\sigma^{(7/\mu_3)}} + \frac{1176\mu_1^5 I_1}{\hat{C}^2\mu_3^2\sigma^{(7/\mu_3)}} + \frac{1372\mu_1^5(2\mu_1-\mu_3)I_2}{\hat{C}^2\mu_3^3\sigma^{\frac{(6\mu_3-14\mu_1)}{\mu_3^2}}}$$
$$+\frac{230\,496\mu_1^7}{\hat{C}^3\mu_3^2(14\mu_1-13\mu_3)\sigma^{(7/\mu_3)}}\log\sigma \tag{5.13}$$

where $I_1$, $I_2$ are integration constants. Substituting equations (5.8), (5.9), (5.12) and (5.13) in (5.5), we get

$$\psi \approx \frac{14\mu_1^2}{\hat{C}\mu_3\sigma^{(2/\mu_3)}} + \eta\left[\frac{I_1}{\sigma^{(2/\mu_3)}} + \frac{I_2}{\sigma^{\frac{(14\mu_1-11\mu_3)}{\mu_3^2}}} + \frac{196\mu_1^2\log\sigma}{\hat{C}(14\mu_1-13\mu_3)\sigma^{(2/\mu_3)}}\right] \tag{5.14}$$

$$\chi \approx \frac{5488\mu_1^7}{\hat{C}^3\mu_3^2\sigma^{(7/\mu_3)}} + \eta\left[\frac{10\,976(3\mu_3-7\mu_1)\mu_1^7}{\hat{C}^3\mu_3^2(14\mu_1-13\mu_3)\sigma^{(7/\mu_3)}} + \frac{1176\mu_1^5 I_1}{\hat{C}^2\mu_3^2\sigma^{(7/\mu_3)}}\right.$$
$$\left. + \frac{1372\mu_1^5(2\mu_1-\mu_3)I_2}{\hat{C}^2\mu_3^3\sigma^{\frac{(6\mu_3-14\mu_1)}{\mu_3^2}}} + \frac{230\,496\mu_1^7\log\sigma}{\hat{C}^3\mu_3^2(14\mu_1-13\mu_3)\sigma^{(7/\mu_3)}}\right]. \tag{5.15}$$

Rewriting in terms of old variables equations (5.14) and (5.15) become

$$u \approx \frac{14\mu_1^2(\mu_3 t+\mu_4)}{\hat{C}\mu_3(\mu_1 x+\mu_2)^2} + z\left[\frac{I_1}{(\mu_1 x+\mu_2)^2} + \frac{I_2(\mu_3 t+\mu_4)^{\frac{\mu_1(14\mu_1-13\mu_3)}{\mu_3^2}}}{(\mu_1 x+\mu_2)^{\frac{(14\mu_1-11\mu_3)}{\mu_3}}}\right.$$
$$\left.+\frac{196\mu_1^2}{\hat{C}(14\mu_1-13\mu_3)(\mu_1 x+\mu_2)^2}\log\left(\frac{(\mu_1 x+\mu_2)^{\mu_3}}{(\mu_3 t+\mu_4)^{\mu_1}}\right)\right] \tag{5.16}$$

$$v \approx \frac{5488\mu_1^7(\mu_3 t+\mu_4)^3}{\hat{C}^3\mu_3^3(\mu_1 x+\mu_2)^7} + z\left[\frac{10\,976(3\mu_3-7\mu_1)\mu_1^7(\mu_3 t+\mu_4)^2}{\hat{C}^3\mu_3^2(14\mu_1-13\mu_3)(\mu_1 x+\mu_2)^7}\right.$$
$$+\frac{1176\mu_1^5 I_1(\mu_3 t+\mu_4)^2}{\hat{C}^2\mu_3^2(\mu_1 x+\mu_2)^7} + \frac{1372\mu_1^5(2\mu_1-\mu_3)I_2(\mu_1 x+\mu_2)^{\frac{(6\mu_3-14\mu_1)}{\mu_3^2}}}{\hat{C}^2\mu_3^3(\mu_3 t+\mu_4)^{\frac{(13\mu_1\mu_3-14\mu_1^2-2\mu_3^2)}{\mu_3^2}}}$$
$$\left.+\frac{230\,496\mu_1^7(\mu_3 t+\mu_4)^2}{\hat{C}^3\mu_3^2(14\mu_1-13\mu_3)(\mu_1 x+\mu_2)^7}\log\left(\frac{(\mu_1 x+\mu_2)^{\mu_3}}{(\mu_3 t+\mu_4)^{\mu_1}}\right)\right]. \tag{5.17}$$

We wish to mention that by using the projectable generator we have obtained in such a way that only one of the new variables, $\sigma$ or $\eta$, depends on the small parameter $z$ (in our case $\eta$) and, moreover, that $\eta$ is an infinitesimal of the same order in $z$.

Taking into account the above remarks in what follows we give a formal procedure (summarized in three steps) to obtain approximate solutions for an $m \times m$ system of PDE's involving two independent variables and a small parameter, $z$, of the form

$$H^i(t, x, z, u^j, u_t^j, u_x^j, \ldots) = 0 \qquad i, j = 1, \ldots, m. \tag{5.18}$$

(1) First choose an appropriate equivalence generator (for instance, a projectable equivalence generator) and reducing the above system $H_i$ to an RS of equations

$$K^i(\sigma, \eta, \chi^j, \chi_\sigma^j, \chi_\eta^j, \ldots) = 0 \qquad i, j = 1, \ldots, m \tag{5.19}$$

such that only one of the variables $\sigma$ or $\eta$ depends on $z$ (in the following we assume $\eta$).

(2) Expand $\chi^j$, *iff* $\eta$ *is an infinitesimal of order greater than or the same as* $z$, in series with respect to $\eta$, such that

$$\chi^j = \chi_0^j(\sigma) + \eta\chi_1^j(\sigma) + \eta^2\chi_2^j(\sigma) + \cdots \qquad j = 1, \ldots, m. \tag{5.20}$$

(3) Truncate (5.20) at order $n$ and substitute in the RS (5.19). By equating the various powers of $\eta^i, i = 0, \ldots, n$ to zero we get a system of ODEs with the independent variable $\sigma$. As shown earlier, only the subsystem ($m$ equations) of zeroth-order equations, that is the system in the unknowns $\chi_o^j$'s, is nonlinear and it is decoupled from the other $m \times n$ equations. Once a particular solution associated with the nonlinear system of equations is found then the remaining equations can be reduced to a linear form which can also be solved (in principle). Substituting all the explicit forms of $\chi^j$'s in equation (5.20) and rewriting in terms of old variables one can get an approximate solution for the PDE (5.18).

## 6. The case $\mu_1 = \mu_3 = 0$

In this case, we get the following similarity variables

$$\sigma = x - (\mu_2/\mu_4)t \qquad \eta = z \tag{6.1}$$

and

$$u = \psi(\sigma, \eta) \qquad v = \chi(\sigma, \eta). \tag{6.2}$$

As described in the previous section, substituting equations (6.1) and (6.2) in equation (4.2) with the first-order approximation of $\eta$ and equating the coefficients of $\eta^i, i = 0, 1$ to zero in the resultant equation, by neglecting the higher powers of $\eta$, we get

$$\frac{\mu_2}{\mu_4}\psi_0^3\psi_0' - \frac{\chi_0'}{\hat{C}} = 0 \tag{6.3a}$$

$$\psi_0^2\psi_0' + \chi_0 = 0 \tag{6.3b}$$

$$\frac{3\mu_2}{\mu_4}\psi_0^2\psi_1\psi_0' + \frac{\mu_2}{\mu_4}\psi_0^3\psi_1' - \frac{\chi_1'}{\hat{C}} = 0 \tag{6.3c}$$

$$\psi_0^6\psi_1' + 6\psi_0^5\psi_1\psi_0' + \frac{2}{\hat{C}}\chi_0\chi_0' - \frac{\mu_2}{\mu_4}\psi_0^4\chi_0' + \psi_0^4\chi_1 + 4\psi_0^3\psi_1\chi_0 = 0. \tag{6.3d}$$

An interesting feature of this case is that one can easily verify that the unperturbed part, equations (6.3*a*,*b*), admits a trivial solution

$$\psi_0 = \text{constant} = A_0 \qquad \chi_0 = 0. \tag{6.4}$$

In other words we get $u = \text{constant} = A_0$ and $v = 0$ which is the well known equilibrium solution for this problem. However, this solution cannot be obtained from the case considered in the previous section. In this case by solving equations (6.3$c, d$), we get

$$\chi_1 = I_1 A_0^2 \exp\left[-\frac{\mu_2 \hat{C} A_0 \sigma}{\mu_4}\right] \tag{6.5}$$

$$\psi_1 = \frac{\mu_4 I_1}{\mu_2 A_0 \hat{C}} \exp\left[-\frac{\mu_2 \hat{C} A_0 \sigma}{\mu_4}\right] + I_2 \tag{6.6}$$

where $I_1$ and $I_2$ are integration constants, so that one can write an approximate solution of the form

$$u \approx A_0 + z\left[\frac{\mu_4 I_1}{\mu_2 A_0 \hat{C}} \exp\left[-\frac{\mu_2 \hat{C} A_0 \sigma}{\mu_4}\right] + I_2\right] \tag{6.7}$$

$$v \approx z\left[I_1 A_0^2 \exp\left[-\frac{\mu_2 \hat{C} A_0 \sigma}{\mu_4}\right]\right]. \tag{6.8}$$

However, one can also find a more general approximate solution of equation (6.3) by following the steps given in case 1, which turn out to be

$$\psi_0 = \frac{4\mu_4}{\mu_2(\hat{C}\sigma + I_1)} \tag{6.9a}$$

$$\chi_0 = \frac{64\hat{C}\mu_4^3}{\mu_2^3(\hat{C}\sigma + I_1)^4} \tag{6.9b}$$

$$\psi_1 = I_2(\hat{C}\sigma + I_1)^3 + \frac{I_3}{(\hat{C}\sigma + I_1)^2} - \frac{8\hat{C}}{(\hat{C}\sigma + I_1)^2}\log(\hat{C}\sigma + I_1) \tag{6.9c}$$

$$\chi_1 = \frac{-16 I_2 \hat{C}\mu_4^2}{\mu_2^2} + \frac{64 I_3 \hat{C}\mu_4^2}{\mu_2^2(\hat{C}\sigma + I_1)^5} - \frac{512\mu_4^2\hat{C}^2}{\mu_2^2(\hat{C}\sigma + I_1)^5}\log(\hat{C}\sigma + I_1) \tag{6.9d}$$

where $I_1$, $I_2$ and $I_3$ are integration constants. Substituting equation (6.8) in (6.2), one obtains an approximate solution of the form

$$\begin{aligned} u \approx{}& \frac{4\mu_4}{\mu_2(\hat{C}(x - \frac{\mu_2}{\mu_4}t) + I_1)} + z\left[I_2\left(\hat{C}\left(x - \frac{\mu_2}{\mu_4}t\right) + I_1\right)^3 + \frac{I_3}{(\hat{C}(x - \frac{\mu_2}{\mu_4}t) + I_1)^2}\right. \\ &\left. - \frac{8\hat{C}\log(\hat{C}(x - \frac{\mu_2}{\mu_4}t) + I_1)}{(\hat{C}(x - \frac{\mu_2}{\mu_4}t) + I_1)^2}\right] \end{aligned} \tag{6.10}$$

$$\begin{aligned} v \approx{}& \frac{64\hat{C}\mu_4^3}{\mu_2^3(\hat{C}(x - \frac{\mu_2}{\mu_4}t) + I_1)^4} + \frac{z\mu_4^2}{\mu_2^2}\left[-16 I_2\hat{C} + \frac{64 I_3\hat{C}}{(\hat{C}(x - \frac{\mu_2}{\mu_4}t) + I_1)^5}\right. \\ &\left. - \frac{512\hat{C}^2\log(\hat{C}(x - \frac{\mu_2}{\mu_4}t) + I_1)}{(\hat{C}(x - \frac{\mu_2}{\mu_4}t) + I_1)^5}\right]. \end{aligned} \tag{6.11}$$

## 7. Conclusions

In this paper we have shown that one can deduce approximate solutions for systems depending on a small parameter through the equivalence transformations. We have utilized the equivalence transformations, which have previously been studied for differential equations, where its constitutive elements are arbitrary functions, for the differential equations containing a small parameter.

In order to illustrate our method for a physical example we have considered a hyperbolic model for heat conduction containing an arbitrary parameter representing the relaxation time of the conductor and obtained approximate solutions.

The procedure we used is based essentially on a change of variables induced from equivalence transformations, which change the system under consideration into another system of PDEs, the so-called RS, with different dependent and independent variables. From the RS one can construct approximate solutions by appropriately expanding the new dependent variables and solving the approximate system. One can, in principle, obtain approximate solutions for other physical models involving a small parameter by following the procedure given in section 5. This method is straightforward and involves only minor calculations compared with other methods based on approximate transformation group analysis.

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